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# SPERNER FAMILIES, s-SYSTEMS, AND A THEOREM OF MESHALKIN

Murray Hochberg and Warren M. Hirsch

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## ABSTRACT

A map $\delta$ from the first r natural numbers to the power set of a set $\Omega$ of cardinality n is called an "r-decomposition" if $\delta(i) \cap \delta(j) = \emptyset$, $i \neq j$, and $\bigcup \delta(i) = \Omega$. A set $\sigma$ of r-decompositions is an "s-system of order r" if, for each index i, no two sets in the set $\{\delta(i): \delta \in \sigma\}$ satisfy an inclusion relation. Meshalkin proved that the cardinality of $\sigma$ cannot exceed M*, where

$$M^* = \max \{\frac{n!}{n_1!n_2!\cdots n_r!} : \sum_1^r n_i = n\} .$$

This generalizes an earlier result of Sperner, who had studied the case r = 2. Although the bound M* is sharp, by analyzing the structure of s-systems of order 2, we indicate how it can be refined. Our main theorem settles the uniqueness question for maximal s-systems, showing that there is an essentially unique s-system containing M* elements and characterizing this system. We give also a new proof of Meshalkin's theorem.

## INTRODUCTION

The problems studied in this paper have evolved from ideas developed by Emanuel Sperner in his very interesting paper "Ein Satz über Untermengen einer endlichen Menge" [5]. The mise-en-scène of Sperner's work is a finite set $\Omega$. A non-empty class $\Phi$ of subsets of $\Omega$ is called a "Sperner family"[1] if no set in $\Phi$ is a subset of any other set in $\Phi$. The number of sets in $\Phi$ is called the "size" of $\Phi$ and is denoted by $|\Phi|$.[2] A Sperner family $\Phi^*$ is called a "maximal family" if there does not exist a Sperner family $\Phi$ whose size exceeds that of $\Phi^*$. Since there are only a finite number of Sperner families that can be formed from the points of $\Omega$, it is plain that there is at least one maximal family. Sperner proved that there is essentially only one maximal family and determined its size and structure. His theorem is the following:

SPERNER'S THEOREM. *If $\Phi$ is a Sperner family of subsets of $\Omega$, $|\Omega| = n$, then $|\Phi| \leq \binom{n}{[n/2]}$.*[3] *Moreover, $\Phi$ is a Sperner family of size $\binom{n}{[n/2]}$ if and only if*

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[1] Our translation of Sperner's term, "ausgezeichnet".

[2] The cardinality of any set A will be denoted by $|A|$.

[3] We use the following standard notation:
$[x]$ is the integer x such that $x-1 < [x] \leq x$ ;
$\binom{n}{k}$ is the binomial coefficient $n!/k!(n-k)!$ .

(a) *when n is even, $\Phi$ consists of all subsets of $\Omega$ of cardinality $n/2$;*

(b) *when n is odd, $\Phi$ consists either of all subsets of $\Omega$ of cardinality $(n-1)/2$ or of all subsets of cardinality $(n+1)/2$.*

With a view to generalizing this theorem it is instructive to formulate it in a slightly different way. Suppose that we associate to each set A in a Sperner family $\Phi$ the set $\Omega - A = A^c$.[4] We obtain in this way a family of ordered pairs, $\{(A,A^c): A \in \Phi\}$,[5] with the following properties:

(i) Each ordered pair is a partition of the space $\Omega$.

(ii) The family of complementary sets, $\{A^c: A \in \Phi\}$, is a Sperner family.

Accordingly, Sperner's Theorem asserts that there are at most $\binom{n}{[n/2]}$ ordered pairs of sets such that each ordered pair is a partition of $\Omega$ and the families of first and second components, respectively, are Sperner families. In the light of this result it seems natural to pose the question formulated below: Let $2^\Omega$ denote the power set of $\Omega$. A map $\delta$: $\{1,2,\cdots,r\} \to 2^\Omega$ such that $\delta(i) \cap \delta(j) = \emptyset$, $i \neq j$, and $\bigcup_{i=1}^{r} \delta(i) = \Omega$ will be called an "r-decomposition" of $\Omega$. Consider a set $\sigma$ of r-decompositions of $\Omega$. Put

$$\Gamma_i = \{\delta(i) : \delta \in \sigma\}, \quad i = 1,2,\cdots,r .$$

(Naturally, if $\delta \in \sigma$, $\zeta \in \sigma$ and $\delta(i) = \zeta(i)$, this set appears only once in $\Gamma_i$.) We shall call $\sigma$ an "s-system of order r" if, for each i, $\Gamma_i$ is a Sperner family.[6] The size of $\sigma$ is, by definition, $|\sigma|$, and we say that $\sigma$ is "based

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[4] We use the superscript "c" to denote complementation. In the sequel, the space relative to which the complement is taken will be clear from the context.

[5] We use the usual notation $\{x: \cdots\}$ to denote the set of elements x such that $\cdots$.

[6] Evidently Sperner families can be identified with s-systems of order 2.

on $\Omega$". What is the maximum size of an s-system of order r?[7]

The problem was solved by L. D. Meshalkin [4] who proved the following theorem:

> MESHALKIN'S THEOREM. *If* $\sigma$ *is an* s-*system of order* r *based on* $\Omega$, $|\Omega| = n$, *then* $|\sigma| \leq M^*$, *where*
>
> $$M^* = \max \{\frac{n!}{n_1!n_2!\cdots n_r!} : \sum_{i=1}^{r} n_i = n\} .$$
>
> *Moreover, there exists at least one* s-*system containing* $M^*$ *elements (called a maximal* s-*system).*

Meshalkin left unresolved the question of whether or not there is an essentially unique maximal s-system, and, if so, how its structure is characterized. These are the principal problems discussed in the sequel.

In Section 1 we give a short, independent proof of Meshalkin's theorem. Although the upper bound $M^*$ is sharp in the sense that it is attained, we indicate in Theorem 2 (Section 2) how it can be refined. The refinement hinges on an analysis of the structure of Sperner families, given in Theorem 1 (Section 2). The main result, Theorem 3 (Section 3), settles the uniqueness question for maximal s-systems, showing that there is an essentially unique s-system containing $M^*$ elements and characterizing this system. Specialized to the case $r = 2$, the proof of Theorem 3 provides a new argument for the uniqueness assertion in Sperner's theorem; although this argument resembles Sperner's in spirit, it differs from his in detail.

## 1. PROOF OF MESHALKIN'S THEOREM

Meshalkin's proof of his theorem depends on estimates similar to those used by Sperner for the special case $r = 2$. In this section we give a proof along completely different

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[7] This question is attributed by L. D. Meshalkin [4] to B. A. Sevast'yanov in tribute to whom the terminology "s-system" has been adopted.

lines, motivated by ideas due to E. Levine [2] and D. Lubell [3]. Throughout the argument $\sigma$ denotes an arbitrary s-system based on $\Omega$, where, for simplicity of notation, we put $\Omega = \{1,2,\cdots,n\}$.

Let $T$ denote the set of all permutations $\pi: \Omega \to \Omega$. We use the conventional symbol "$\{\pi(j)\}$" for the singleton set whose only member is $\pi(j)$. To each r-decomposition $\delta \in \sigma$ we associate r+1 sums, $S_i(\delta)$, $i = 0,1,\ldots,r$, and a set $E_\delta \subset T$, defined by the equations

$$S_i(\delta) = \begin{cases} 0 & , \quad i = 0 , \\ \sum_{j=1}^{i} |\delta(j)| & , \quad i = 1,2,\ldots,r , \end{cases}$$

and

$$E_\delta = \{\pi \in T : \delta(i) = \bigcup_{j=S_{i-1}(\delta)+1}^{S_i(\delta)} \{\pi(j)\},\ i=1,2,\ldots,r\}.$$

It is evident that

$$|E_\delta| = \prod_{i=1}^{r} |\delta(i)|! \ .$$

Suppose that $\delta$ and $\zeta$ are two r-decompositions in $\sigma$ such that $E_\delta \cap E_\zeta \neq \emptyset$. Let $\pi_0 \in E_\delta \cap E_\zeta$. Since

$$\delta(1) = \bigcup_{j=1}^{S_1(\delta)} \{\pi_0(j)\},$$

$$\zeta(1) = \bigcup_{j=1}^{S_1(\zeta)} \{\pi_0(j)\},$$

and $\delta(1)$, $\zeta(1)$ are by definition elements of a Sperner family, we must have $S_1(\delta) = S_1(\zeta)$ and $\delta(1) = \zeta(1)$. By induction it follows easily that $\delta(i) = \zeta(i)$, $i=1,2,\ldots,r$. Thus, if $\delta$ and $\zeta$ are distinct r-decompositions in $\sigma$,

$$E_\delta \cap E_\zeta = \emptyset \ .$$

Since $\bigcup_{\delta \in \sigma} E_\delta \subset T$ and $|T| = n!$, it follows that

$$(1) \qquad \Big|\bigcup_{\delta \in \sigma} E_\delta\Big| = \sum_{\delta \in \sigma} |E_\delta| = \sum_{\delta \in \sigma} \prod_{i=1}^{r} |\delta(i)|! \leq n! \ .$$

Finally, we note that since $\sum_{i=1}^{r} |\delta_i| = n$, and $M^*$ is the maximal multinomial coefficient,

$$\frac{M^*}{n!} \prod_{i=1}^{r} |\delta(i)|! \geq 1 .$$

Hence, from (1),

$$|\sigma| = \sum_{\delta \in \sigma} 1 \leq \frac{M^*}{n!} \sum_{\delta \in \sigma} \prod_{i=1}^{r} |\delta(i)|! \leq M^* .$$

This establishes the Meshalkin bound for the size of an s-system.

To construct an s-system of size $M^*$ we set $q = [n/r]$, $\rho = n - rq$, and let $\sigma^*$ denote the set of all r-decompositions $\delta$ satisfying the conditions, $|\delta(i)| = q+1$, $i=1,2,\ldots,\rho$, and $|\delta(i)| = q$, $i = \rho+1,\rho+2,\ldots,r$. Since distinct sets of the same cardinality cannot satisfy an inclusion relation, it is evident that $\sigma^*$ is an s-system. By an elementary argument its size is given by

$$|\sigma^*| = \frac{n!}{[(q+1)!]^{\rho}[q!]^{r-\rho}} ,$$

which, as is well known, is the maximum multinomial coefficient, $M^*$.[8]

## 2. REFINEMENT OF MESHALKIN'S BOUND

We have seen that the Meshalkin bound is sharp. Nevertheless, as we shall demonstrate in this section, it can be improved by classifying the set of all s-systems into certain natural categories and giving for each of these (except the maximal systems) a more precise bound than $M^*$. We begin with an elementary but basic observation:

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[8] For a proof that the maximum multinomial coefficient is $n!/[(q+1)!]^{\rho}[q!]^{r-\rho}$, see G. Chrystal, Algebra, Part II, Sixth Edition, (New York: Chelsea Publishing Co., 1952), pp. 16-17.

LEMMA 1. *Let $\Lambda$ be a non-empty collection of subsets of $\Omega$ such that each set in $\Lambda$ has cardinality $k$, $1 < k < n$, and $|\Lambda| < \binom{n}{k}$. Then there exist sets $A \in \Lambda$ and $A^* \in \Lambda^c$ such that $|A^*| = k$ and $|A \cap A^*| = k-1$.*

Proof. Let $A_0$ be an arbitrary set in $\Lambda$. To simplify notation we put $\Omega = \{1,2,\ldots,n\}$ and $A_0 = \{1,2,\ldots,k\}$. Since $|\Lambda| < \binom{n}{k}$, there exists a set $B \in \Lambda^c$ such that $|B| = k$; without loss of generality we may assume (by relabeling elements if necessary) that B has the form,

$$B = \{1,2,\ldots,m,i_{m+1},i_{m+2},\ldots,i_k\} ,$$

where $0 \leq m < k$ and $i_j \in A^c$, $j = m+1,m+2,\ldots,k$. We define recursively a finite sequence of sets by setting

$$A_\nu = (A_{\nu-1} - \{m+\nu\}) \cup \{i_{m+\nu}\} , \quad \nu = 1,2,\ldots,k-m.$$

Plainly, $|A_\nu| = k$, $|A_{\nu-1} \cap A_\nu| = k-1$, $\nu = 1,2,\ldots,k-m$, and $A_{k-m} = B$. Since the first set in the sequence $A_0,A_1,\ldots,A_{k-m}$ is in $\Lambda$ and the last set is in $\Lambda^c$, there exist consecutive sets $A_{\nu_0-1}$, $A_{\nu_0}$ such that $A_{\nu_0-1} \in \Lambda$ and $A_{\nu_0} \in \Lambda^c$; these can be taken to be, respectively, the sets $A$ and $A^*$ of the lemma.

Let $A \subset \Omega$ be an arbitrary set. By an "extension of A" is meant any set $B \subset \Omega$ such that $A \subset B$ and $|B| = |A| + 1$. If $\Lambda$ is a non-empty set of subsets of $\Omega$, we shall denote by $\Lambda^+$ the set of all extensions of sets in $\Lambda$; if $\Lambda = \emptyset$, we define $\Lambda^+ = \emptyset$.[9] For the theorems to follow we need an estimate of $|\Lambda^+|$ in a special case, which is provided by the next lemma.

LEMMA 2. *Let $\Lambda$ be a collection of subsets of $\Omega$ such that each set in $\Lambda$ has cardinality $k$, $0 \leq k \leq n$.*[10] *Then*

---

[9] Note that if $\Lambda = \{\emptyset\}$, then $\Lambda^+ = \{A \subset \Omega : |A| = 1\}$.

[10] If $\Lambda = \emptyset$, each set in $\Lambda$ vacuously has cardinality k for any integer k.

(2) $$|\Lambda^+| \geq \frac{|\Lambda|(n-k)}{k+1} ,$$

and for $\Lambda \neq \emptyset$ equality holds if and only if $|\Lambda| = \binom{n}{k}$.

Proof. Since (2) holds trivially when either $\Lambda = \emptyset$ or $k = n$, we may assume that $\Lambda \neq \emptyset$ and $k < n$. Consider the set of ordered pairs,

$$E = \{(A,B): A \in \Lambda, B \in \Lambda^+\} .$$

Since each set $A \in \Lambda$ has precisely $n-k$ extensions,

$$|E| = |\Lambda|(n-k) .$$

If we now set

$$F = \{(A,B): B \in \Lambda^+, A \subset B, |A| = k\} ,$$

it is clear that $E \subset F$ and $|F| = (k+1)|\Lambda^+|$. Hence,

$$(k+1)|\Lambda^+| \geq |\Lambda|(n-k) ,$$

which is equivalent to (2).

If $|\Lambda| = \binom{n}{k}$, then

$$\binom{n}{k+1} \geq |\Lambda^+| \geq \frac{|\Lambda|(n-k)}{k+1} = \binom{n}{k+1} ,$$

so that equality holds in (2).

Suppose conversely that $\Lambda \neq \emptyset$ and

(3) $$|\Lambda^+| = \frac{|\Lambda|(n-k)}{k+1} .$$

We wish to prove that $|\Lambda| = \binom{n}{k}$; since this is trivial for the cases $k = 0$ and $k = n$, we may restrict attention to integers $k$ such that $0 < k < n$. If $|\Lambda| < \binom{n}{k}$, by Lemma 1 there are sets $A \in \Lambda$, $A^* \in \Lambda^c$ such that $|A^*| = k$ and $|A \cap A^*| = k-1$. Put

$$B = A \cup A^* .$$

Then $B \in \Lambda^+$ and

(4) $$(A^*,B) \in E^c \cap F .$$

On the other hand, (3) implies that

$$|F| = (k+1)|A^+| = |A|(n-k) = |E| .$$

Since $E \subset F$, we conclude that $E = F$, which contradicts (4) and completes the proof.

Let $\Phi$ be a Sperner family. We set

$$\Gamma_i = \{A \in \Phi : |A| = i\} , \quad i = 0,1,\ldots,n,$$

and

$$\gamma_i = |\Gamma_i| .$$

We shall refer to the vector $(\gamma_0,\gamma_1,\ldots,\gamma_n)$ as the "fine structure" of $\Phi$. We define recursively numbers $r_i$, $i = 1,2,\ldots,n$, related to the fine structure by putting

$$\Lambda_0 = \Gamma_0 ,$$

$$\Lambda_i = \Lambda_{i-1}^+ \cup \Gamma_i , \quad i = 1,2,\ldots,n,$$

and

$$r_i = |\Lambda_{i-1}^+| , \quad i = 1,2,\ldots,n.$$

Since $\Phi$ is a Sperner family, $\Lambda_{i-1}^+ \cap \Gamma_i = \emptyset$, $i=1,2,\ldots,n$, and hence

$$|\Lambda_i| = |\Lambda_{i-1}^+| + |\Gamma_i| = r_i + \gamma_i . \tag{5}$$

Although the formal definition of $r_i$ suffices for our purposes, it is useful in interpreting the next theorem to observe that

$$r_i = |\{B \subset \Omega: |B| = i, \ \exists\, A \in \Phi \text{ such that } A \subset B\}| .$$

We state now a fundamental property of the fine structure:

THEOREM 1. *The fine structure $(\gamma_0,\gamma_1,\ldots,\gamma_n)$ of an arbitrary Sperner family $\Phi$ satisfies the inequalities*

$$\sum_{j=0}^{i-1} \frac{\gamma_j}{\binom{n}{j}} \le \frac{r_i}{\binom{n}{i}} , \quad i = 1,2,\ldots,n. \tag{6}$$

*Moreover, if the inequality is strict for some index $i_0$, it is strict for all $i > i_0$.*

Proof. We argue inductively. It is evident that either $\gamma_0 = 0$ or $\gamma_0 = 1$, and in the latter case, $r_1 = n$. Thus, (6) holds for $i = 1$. Assume now that (6) is satisfied when $i = k < n$. Using (5) and the inductive hypothesis we obtain the inequality,

$$(7) \qquad \sum_{j=0}^{k} \frac{\gamma_j}{\binom{n}{j}} = \sum_{j=0}^{k-1} \frac{\gamma_j}{\binom{n}{j}} + \frac{\gamma_k}{\binom{n}{k}} \leq \frac{r_k+\gamma_k}{\binom{n}{k}} = \frac{|\Lambda_k|}{\binom{n}{k}} .$$

Since each set in $\Lambda_k$ has cardinality k, it follows from Lemma 2 that

$$(8) \qquad \sum_{j=0}^{k} \frac{\gamma_j}{\binom{n}{j}} \leq \frac{|\Lambda_k^+|(k+1)}{\binom{n}{k}(n-k)} = \frac{r_{k+1}(k+1)}{\binom{n}{k}(n-k)} = \frac{r_{k+1}}{\binom{n}{k+1}} ,$$

which completes the induction.

If the set of indices i for which (6) is strict contains $i_0 < n$, then (7) and (8) show that it also contains $i_0 + 1$. This completes the proof of Theorem 1.

COROLLARY. The fine structure of a Sperner family satisfies the inequality

$$(9) \qquad \sum_{j=0}^{n} \frac{\gamma_j}{\binom{n}{j}} \leq 1 ,$$

with equality if and only if there is an integer k such that $\gamma_k = \binom{n}{k}$.

Proof. If $\Gamma_n \neq \emptyset$, then $\gamma_n = 1$ and $\gamma_j = 0$ for $j < n$, so that (9) holds. If $\Gamma_n = \emptyset$, then $\Gamma_i \neq \emptyset$ for some index $i<n$, since, by definition, a Sperner family is non-empty. It follows that $\Lambda_{n-1} \neq \emptyset$, which in turn implies that $r_n = |\Lambda_{n-1}^+| = 1$. Setting $i = n$ in Theorem 1 and noting that $\gamma_n = 0$, we obtain (9).

If there is an integer k such that $\gamma_k = \binom{n}{k}$, we shall say that the Sperner family $\Phi$ is "k-saturated" or that the "saturation condition is satisfied". It is easily seen that satisfaction of the saturation condition is necessary and sufficient for equality in (9). Sufficiency follows immediately from the observation that if $\Phi$ is k-saturated. then $\gamma_j = 0$, $j \neq k$.

To prove necessity suppose that $\Phi$ is not k-saturated for any index k. Since $\Phi \neq \emptyset$, there is an index $k < n$ such that $\Gamma_j = \emptyset$, $j < k$, and $\Gamma_k \neq \emptyset$. Then $\Lambda_k = \Gamma_k$ and, accordingly, $0 < |\Lambda_k| < \binom{n}{k}$. By Lemma 2 we conclude that

$$|\Lambda_k| < \frac{|\Lambda_k^+|(k+1)}{n-k} ,$$

and hence

$$\sum_{j=0}^{k} \frac{\gamma_j}{\binom{n}{j}} = \frac{|\Lambda_k|}{\binom{n}{k}} < \frac{|\Lambda_k^+|(k+1)}{\binom{n}{k}(n-k)} = \frac{r_{k+1}}{\binom{n}{k+1}} .$$

By Theorem 1,

$$\sum_{j=0}^{n} \frac{\gamma_j}{\binom{n}{j}} = \sum_{j=0}^{n-1} \frac{\gamma_j}{\binom{n}{j}} < r_n = 1 ,$$

where we have used the facts that $\gamma_n = 0$ and $r_n = 1$, since $\Phi$ is not n-saturated.

Each s-system $\sigma$ of order r has associated with it r Sperner families, $\{\delta(i): \delta \in \sigma\}$, $i = 1,2,\ldots,r$, and by classifying all s-systems according to the fine structures of their associated Sperner families, we are led to the following refinement of the Meshalkin bound:

THEOREM 2. *Let $\sigma$ be an s-system of order r based on $\Omega$, $|\Omega| = n$, and let $\{\delta(i): \delta \in \sigma\}$, $i = 1,2,\ldots,r$, be its associated Sperner families. Let $(\gamma_{i0},\gamma_{i1},\ldots,\gamma_{in})$ denote the fine structure of the Sperner family $\{\delta(i): \delta \in \sigma\}$. Then*

$$(10) \qquad \frac{|\sigma|}{\min\limits_{1 \le i \le r} \sum\limits_{j=0}^{n} \frac{\gamma_{ij}}{\binom{n}{j}}} \le M^* .$$

*Proof.* Let i denote an arbitrary integer, $1 \le i \le r$. Since i will be held fixed throughout our argument, we shall not display explicitly in our notation dependence on i. For example, we write $(\gamma_0,\gamma_1,\ldots,\gamma_n)$ in place of $(\gamma_{i0},\gamma_{i1},\ldots,\gamma_{in})$, where $\gamma_j$ is the cardinality of the set $\Gamma_j$,

$$\Gamma_j = \{\delta(i): \delta \in \sigma , |\delta(i)| = j\} .$$

Let $A \in \Gamma_j$. We define

$$\Delta_A = \{\delta \in \sigma : \delta(i) = A\} .$$

Then, trivially,

$$\{\delta \in \sigma : |\delta(i)| = j\} = \bigcup_{A \in \Gamma_j} \Delta_A ,$$

the union being disjoint. Setting

$$\beta_j = |\{\delta \in \sigma : |\delta(i)| = j\}| ,$$

it follows that

$$|\sigma| = \sum_{j=0}^{n} \beta_j = \sum_{j=0}^{n} \sum_{A \in \Gamma_j} |\Delta_A| .$$

For fixed $A \in \Gamma_j$ we estimate $|\Delta_A|$ by defining a map $\delta \to \bar{\delta}$ on $\Delta_A$, where $\bar{\delta} : \{1,2,\ldots,r-1\} \to \Omega - A$ is the function defined by

$$\bar{\delta}(k) = \begin{cases} \delta(k) , & k = 1,2,\ldots,i-1, \\ \delta(k+1) , & k = i,i+1,\ldots,r-1 . \end{cases}$$

Since $\delta$ is an $r$-decomposition of $\Omega$ and $\delta(i) = A$, it is clear that $\bar{\delta}$ is an $(r-1)$-decomposition of $\Omega - A$. Moreover, since $\sigma$ is an $s$-system, the set $\{\bar{\delta} : \delta \in \Delta_A\}$ is also an $s$-system; its order is $r-1$ and it is based on $\Omega - A$. If $\delta \in \Delta_A$, $\zeta \in \Delta_A$, $\delta \neq \zeta$, then since $\delta(i) = \zeta(i)$, it is clear that $\bar{\delta} \neq \bar{\zeta}$, which shows that the map $\delta \to \bar{\delta}$ is an injection. Writing $M^*_{r,n}$ in place of $M^*$ to display the dependence of the maximum multinomial coefficient on $r$ and $n$, it follows that

$$|\Delta_A| = |\{\bar{\delta} : \delta \in \Delta_A\}| \leq M^*_{r-1,n-j} .$$

Hence

$$|\sigma| \leq \sum_{j=0}^{n} \gamma_j M^*_{r-1,n-j} .$$

It is easily verified that $\binom{n}{j} M^*_{r-1,n-j} \leq M^*_{r,n}$. Consequently, we obtain the inequality

$$|\sigma| \leq M^*_{r,n} \sum_{j=0}^{n} \frac{\gamma_j}{\binom{n}{j}} . \tag{11}$$

Since (11) holds for arbitrary i, it is equivalent to (10).

## 3. SOLUTION OF THE UNIQUENESS PROBLEM

We saw in Section 2 that a maximal s-system of order r based on $\Omega$, $|\Omega| = n$, is obtained by forming the set $\sigma^*$ of all r-decompositions $\delta$ satisfying the conditions, $|\delta(i)| = q+1$, $i = 1,2,\ldots,\rho$, and $|\delta(i)| = q$, $i = \rho+1,\rho+2,\ldots,r$, where $q = [n/r]$ and $\rho = n - rq$. We shall prove that any maximal s-system has essentially the same structure as $\sigma^*$.

LEMMA 3. *Let $\sigma$ be any s-system such that $|\sigma| = M^*$. To each r-decomposition $\delta \in \sigma$ there corresponds a set $J_\delta \subset \{1,2,\ldots,r\}$ such that $|J_\delta| = \rho$, $|\delta(i)| = q+1$, $i \in J_\delta$, and $|\delta(i)| = q$, $i \in J_\delta^c$.*

*Proof.* In our proof of Meshalkin's Theorem (see Section 1) we showed that

$$\frac{M^*}{n!} \sum_{\delta \in \sigma} \prod_{i=1}^{r} |\delta(i)|! \leq M^* .$$

On the other hand, since $|\sigma| = M^*$ and $\frac{M^*}{n!} \prod_{i=1}^{r} |\delta(i)|! \geq 1$, we have

$$M^* \leq \frac{M^*}{n!} \sum_{\delta \in \sigma} \prod_{i=1}^{r} |\delta(i)|! .$$

It follows that for each $\delta \in \sigma$,

$$\frac{n!}{|\delta(1)|!\ |\delta(2)|! \cdots |\delta(r)|!} = M^* .$$

As is well known[11] this is possible only if there exists a set $J_\delta \subset \{1,2,\ldots,r\}$ such that $|J_\delta| = \rho$, $|\delta(i)| = q+1$, $i \in J_\delta$, and $|\delta(i)| = q$, $i \in J_\delta^c$.

We now obtain a complete characterization of maximal s-systems by proving that the sets $J_\delta$ whose existence was just established are the same for all $\delta \in \sigma$.

[11] Chrystal, op. cit., pp. 16-17.

THEOREM 3. *Let* $\sigma$ *be any s-system such that* $|\sigma| = M^*$. *There exists a set* $J \subset \{1,2,\ldots,r\}$ *such that* $|J| = \rho$ *and for all r-decompositions* $\delta \in \sigma$, $|\delta(i)| = q+1$, $i \in J$, $|\delta(i)| = q$, $i \in J^c$.

*Proof*. Using the notation

$$\Gamma_{ij} = \{\delta(i): \delta \in \sigma, |\delta(i)| = j\}$$

and

$$\gamma_{ij} = |\Gamma_{ij}|,$$

we conclude from Theorem 2 and the Corollary to Theorem 1 that for each index i,

$$\sum_{j=0}^{n} \frac{\gamma_{ij}}{\binom{n}{j}} = 1 .$$

Consequently, there exists an integer $k_i$ such that

$$\gamma_{ik_i} = \binom{n}{k_i} \quad , i = 1,2,\ldots,r.$$

By Lemma 3, either $k_i = q$ or $k_i = q+1$. This implies that for each index i, either $|\delta(i)| = q$ for all $\delta \in \sigma$ or $|\delta(i)| = q+1$ for all $\delta \in \sigma$. Hence, the set

$$J_\delta = \{i: |\delta(i)| = q+1\}$$

does not depend on $\delta$.

Theorem 3 shows that in an obvious sense any maximal s-system $\sigma$ is transformed by a permutation of $\{1,2,\ldots,r\}$ to the maximal s-system $\sigma^*$ defined at the beginning of this section. Thus, modulo permutations, there is a unique maximal s-system.

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13. ABSTRACT A map $\delta$ from the first r natural numbers to the power set of a set $\Omega$ of cardinality n is called an "r-decomposition" if $\delta(i) \cap \delta(j) = \emptyset$, $i \neq j$, and $\cup \delta(i) = \Omega$. A set $\sigma$ of r-decompositions is an "s-system of order r" if, for each index i, no two sets in the set $\{\delta(i) : \delta \in \sigma\}$ satisfy an inclusion relation. Meshalkin proved that the cardinality of $\sigma$ cannot exceed $M^*$, where

$$M^* = \max \left\{ \frac{n!}{n_1! n_2! \ldots n_r!} : \sum_1^r n_i = n \right\} .$$

This generalizes an earlier result of Sperner, who had studied the case r = 2. Although the bound $M^*$ is sharp, by analyzing the structure of s-systems of order 2, we indicate how it can be refined. Our main theorem settles the uniqueness question for maximal s-systems, showing that there is an essentially unique s-system containing $M^*$ elements and characterizing this system. We give also a new proof of Meshalkin's theorem.

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